

Time transfer functions in Schwarzschild-like metrics in the weak-field limit: A unified description of Shapiro and lensing effects

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We present a complete analysis of the light rays within the linearized, weak-field approximation of a Schwarzschild-like metric describing the gravitational field of an isolated, spherically symmetric body. We prove in this context the existence of two time transfer functions and we obtain these functions in an exact closed-form. We are led to distinguish two regimes. In the first regime, the two time transfer functions correspond to rays which are confined in regions of spacetime where the weak-field approximation is valid. Such a regime occurs in gravitational lensing configurations with double images of a given source. We find the general expressions of the angular separation and the difference in light travel time between the two images. In the second regime, there exists only one time transfer function corresponding to a light ray remaining in a region of weak field. Performing a Taylor expansion of this function with respect to the gravitational constant, we obtain the Shapiro time delay completed by a series of so-called “enhanced terms.” The enhanced terms beyond the third order are new.

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I. INTRODUCTION

The notion of time transfer function is of crucial interest for modeling the geometric optical effects in the metric theories of gravity. In order to precise what we mean by a time transfer function in a given spacetime V_4 , consider a light ray Γ propagating in a region of V_4 covered by a single coordinate system $x^\alpha = (x^0, \mathbf{x})$, with $x^0 = ct$ and $\mathbf{x} = (x^i)$, $i = 1, 2, 3$. Let (x_A^0, \mathbf{x}_A) be the point-event where Γ is emitted and (x_B^0, \mathbf{x}_B) the point-event where it is observed. The light travel time $(x_B^0 - x_A^0)/c$ is a function of x_A^0, \mathbf{x}_A and \mathbf{x}_B or a function of \mathbf{x}_A, x_B^0 , and \mathbf{x}_B , each of these functions depending on the ray Γ . So we may write

$$x_B^0 - x_A^0 = c\mathcal{T}_{e,\Gamma}(x_A^0, \mathbf{x}_A, \mathbf{x}_B) = c\mathcal{T}_{r,\Gamma}(\mathbf{x}_A, x_B^0, \mathbf{x}_B). \quad (1)$$

Following a terminology introduced in [1], $\mathcal{T}_{e,\Gamma}$ (resp. $\mathcal{T}_{r,\Gamma}$) is called the (coordinate) emission (resp. reception) time transfer function associated with the light ray Γ . Of course, $\mathcal{T}_{r,\Gamma}$ can be in principle computed if $\mathcal{T}_{e,\Gamma}$ is known, and vice versa. It must be noted that for any stationary metric written in adapted coordinates, the time transfer functions $\mathcal{T}_{e,\Gamma}$ and $\mathcal{T}_{r,\Gamma}$ are one and the same function \mathcal{T}_Γ which depends only upon \mathbf{x}_A and \mathbf{x}_B .

Knowing a time transfer function associated with a light ray enables one to model the range and Doppler

observables of space missions (see, e.g., [2–4] and Refs. therein). It also allows the precise modeling of modern astrometry, since the propagation direction of the light ray Γ can be determined from the first derivatives of $\mathcal{T}_{e,\Gamma}$ or $\mathcal{T}_{r,\Gamma}$ at the point of observation [1, 5, 6]. However, in spite of its practical interest, the explicit determination of the whole set of possible time transfer functions in a given spacetime remains an unsolved problem. Even for the Schwarzschild metric in which the exact solutions to the geodesic equations are known (see, e.g., [7] and Refs. therein), the null geodesics passing through two given points are only determined in an implicit, very hard to work manner [8].

Nevertheless, a procedure enabling one to determine a particular type of time transfer functions has been obtained in [5] for any metric which may be expanded in a power series of the gravitational constant G as follows:

$$g_{\mu\nu}(x^\alpha, G) = \eta_{\mu\nu} + \sum_{n=1}^{\infty} G^n g_{\mu\nu}^{(n)}(x^\alpha), \quad (2)$$

where $\eta_{\mu\nu}$ is the Minkowski metric. This procedure is based on the assumption that there exists a null geodesic linking the emitter and the receiver such that the associated reception time transfer function, say \mathcal{T}_r , is given by an expansion having the form

$$c\mathcal{T}_r(\mathbf{x}_A, x_B^0, \mathbf{x}_B) = |\mathbf{x}_B - \mathbf{x}_A| + c \sum_{n=1}^{\infty} \mathcal{T}_r^{(n)}(\mathbf{x}_A, x_B^0, \mathbf{x}_B), \quad (3)$$

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where $\mathcal{T}_r^{(n)}$ stands for a term of order G^n . One of us has proposed to call such a null geodesic a quasi-Minkowskian light ray (see [9]). It may be shown that each $\mathcal{T}_r^{(n)}$ can be expressed by integrals involving only the terms of order $1, \dots, n-1$ taken along a zeroth-order null straight line passing through \mathbf{x}_A and \mathbf{x}_B . This iterative procedure implies the uniqueness of the quasi-Minkowskian light ray joining \mathbf{x}_A , \mathbf{x}_B for x_B^0 given. It must be pointed out, however, that the existence of such a ray is not ensured, as we shall see later.

A lot of studies at the first order in G have been devoted to the quasi-Minkowskian light rays traveling through the field of moving bodies (see, e.g., [10–15]). For the Schwarzschild-like metrics, $\mathcal{T}^{(1)}$ is given by the well-known Shapiro formula [16, 17]. In this special case, the higher order functions $\mathcal{T}^{(n)}$ have been also explicitly determined up to $n=3$: the expression of $\mathcal{T}^{(2)}$ is henceforth a classical result (see [1, 5, 18], and Refs. therein). The expression of $\mathcal{T}^{(3)}$ has been found in recent works [19, 20].

Recent works have revealed that the approximation of first order in G is not sufficient for modeling all the solar system experiments. The analysis of the Cassini mission data has shown that the Shapiro formula must be supplemented by higher-order terms called “enhanced terms” which become significant for light rays almost grazing the Sun [2]. More recently, a so-called “enhanced 2PN term” has been taken into account for modeling light rays passing near the limb of the Sun [3] or the giant planets of the solar system [9, 18]. It is worth noticing that an additional G^2 -term was already introduced in [21]. However, these higher-order terms are linked to the linear part of the metric perturbation since they only depend on the post-Newtonian parameter γ which characterizes the 1PN curvature of space. So it appears that nonlinear contributions to the light travel time must be taken into account even in the linear, weak-field approximation. This feature shows that the expansion in Eq. (3) is not legitimate when the emitter and the receiver are located in almost diametrically opposite directions.

A similar problem is facing us when calculating the light deflection angle, as it has been already noticed in [22]. It is clear that the above-mentioned methods are not well suited to address the problems raised by the gravitational lensing, where multiple images appear. At least one of the two rays is obviously not a quasi-Minkowskian ray. The aim of this paper is to remedy this confusing situation for a Schwarzschild-like metric treated in the limit of the linearized, weak-field approximation.

The paper is organized as follows. In Sec. II, we specify the linearized, weak-field metric and give the expression of the single time transfer function in the case of the radial light rays. In Sec. III, we find the exact solution to the null geodesic equations. Each ray is a branch of hyperbola skirting round the origin of the polar coordinates and having this origin as a focus. In Sec. IV, we show that there exist two distinct rays passing through two given points and we determine explicitly their im-

port parameters. In Sec. V, we derive the full expression of the two corresponding time transfer functions. In Sec. VI, we show that essentially two regimes are recovered from our unifying treatment. In Sec. VII, we determine the direction of light propagation for each possible ray. In Sec. VIII, we apply our results to the gravitational lensing, which corresponds to the first regime. Analyzing the second regime in Sec. IX, we get the Shapiro time delay completed by a series of enhanced terms. We present concluding remarks in Sec. X.

II. GENERALITIES

Let us consider a static, spherically symmetric space-time corresponding to a central mass M . Assuming the coordinates (x^0, \mathbf{x}) to be quasi-Cartesian isotropic coordinates, the metric may be written in the form

$$ds^2 = \mathcal{A}(r)[(dx^0)^2 - \mathcal{U}(r)\delta_{ij}dx^i dx^j], \quad (4)$$

where $r = |\mathbf{x}|$. The corresponding spherical coordinates (r, ϑ, φ) will be also used, so that

$$\delta_{ij}dx^i dx^j = dr^2 + r^2 d\vartheta^2 + r^2 \sin^2 \vartheta d\varphi^2.$$

It is well known that the null geodesics considered as point sets are identical for two conformally related metrics (see, e.g., [23]). Consequently, the light rays will be treated here as null geodesics of the metric conformally related to (4) defined by

$$d\tilde{s}^2 = (dx^0)^2 - \mathcal{U}(r)\delta_{ij}dx^i dx^j. \quad (5)$$

This metric just involves one gravitational potential \mathcal{U} . Setting $m = GM/c^2$, it is assumed that this gravitational potential can be represented as a power series in m/r at any point exterior to the central mass, that is

$$\mathcal{U}(r) = 1 + 2\kappa_1 \frac{m}{r} + 2 \sum_{n=2}^{\infty} \kappa_n \frac{m^n}{r^n}, \quad (6)$$

where the quantities κ_n are constants linked to the post-Newtonian parameters involved in the metric (4). In particular, one has

$$\kappa_1 = 1 + \gamma. \quad (7)$$

In general relativity, $\kappa_1 = 2$.

Henceforth, we restrict our attention to the linearized, weak-field approximation. This means that the terms of order m^2/r^2 in the metric are neglected, so that Eq. (5) reduces to

$$d\tilde{s}^2 = (dx^0)^2 - \left(1 + 2\kappa_1 \frac{m}{r}\right) \delta_{ij}dx^i dx^j. \quad (8)$$

We study the light rays joining an emitter located at point \mathbf{x}_A and a receiver located at point \mathbf{x}_B , both located at a finite distance from the origin of coordinates.

These light rays are regarded as *exact* solutions to the null geodesics equations of the metric (8). This means that we do not assume from the beginning that the solutions can be represented as power series in m . However, restricting to a linearized metric implies that only the light rays confined in regions of spacetime such that $r \gg m$ can be considered as relevant in the present anal-

ysis. In accordance with this restriction, it will be always assumed that $r_A \gg m$ and $r_B \gg m$ in what follows, with $r_A = |\mathbf{x}_A|$ and $r_B = |\mathbf{x}_B|$.

When the vectors \mathbf{x}_A and \mathbf{x}_B are collinear and have the same direction, the time transfer function \mathcal{T}_{rad} associated with the radial null geodesics passing through \mathbf{x}_A and \mathbf{x}_B is given by an elementary integration:

$$c\mathcal{T}_{rad}(r_A, r_B) = \text{sgn}(r_B - r_A) \left[\sqrt{r_B(r_B + 2\kappa_1 m)} - \sqrt{r_A(r_A + 2\kappa_1 m)} + 2\kappa_1 m \ln \left(\frac{\sqrt{r_B} + \sqrt{r_B + 2\kappa_1 m}}{\sqrt{r_A} + \sqrt{r_A + 2\kappa_1 m}} \right) \right]. \quad (9)$$

Henceforth, we concentrate our attention exclusively on the nonradial light rays.

III. NONRADIAL NULL GEODESICS

Since the metric is spherically symmetric, a nonradial light ray joining \mathbf{x}_A and \mathbf{x}_B is confined to the plane passing through \mathbf{x}_A , \mathbf{x}_B and the origin O of spatial coordinates. We choose the coordinate system so that $\vartheta = \pi/2$ for this plane. Two constants of the motion can then be formed:

$$\frac{dx^0}{d\lambda} = E, \quad (10)$$

$$r^2 \left(1 + 2\kappa_1 \frac{m}{r} \right) \frac{d\varphi}{d\lambda} = J, \quad (11)$$

where λ is an arbitrary affine parameter of the solution. It is easily shown that $d\tilde{s}^2 = 0$ along a null geodesic implies $E \neq 0$. As a consequence, we put

$$b = \frac{J}{E}. \quad (12)$$

Clearly, we have $b \neq 0$ for any nonradial null geodesic. In what follows, b is an algebraic quantity. The absolute value $|b|$ may be interpreted as the impact parameter of the light ray [7]. By convention, we suppose that $E > 0$, which means that the affine parameter λ is increasing with time. So b has the sign of J . As a consequence, $b > 0$ (resp. $b < 0$) when φ is an increasing (resp. a decreasing) function of time.

It follows from the null geodesic equations of the metric (8) that φ and r are linked by the differential equation

$$\left(\frac{d\varphi}{dr} \right)^2 = \frac{b^2}{r^2(r^2 + 2\kappa_1 m r - b^2)}. \quad (13)$$

This equation coincides with the Newtonian differential equation of the trajectory of a fictitious massive particle gravitating with an hyperbolic motion in the field of a pointlike mass $\kappa_1 M$ and having the speed c at infinity [24], a feature which straightforwardly implies that the light deflection predicted by the present approximation

is exactly κ_1 times the Newtonian deflection. Each light ray is therefore a branch of hyperbola of which one of the foci is the origin O , this branch skirting round O . It follows from classical formulas for a Keplerian trajectory that the length of the semitransverse axis a is given by

$$a = \kappa_1 m \quad (14)$$

and the length of its semiconjugate axis coincides with the impact parameter $|b|$.

Denoting by φ_P the value of φ at the pericenter, the polar equation of a light ray solution to Eq. (13) may be written as

$$\frac{1}{r} = \frac{1}{p} [1 + e \cos(\varphi - \varphi_P)], \quad (15)$$

where p is the parameter of the hyperbola and e its eccentricity, with these quantities being given by

$$p = \frac{b^2}{\kappa_1 m}, \quad (16)$$

$$e = \frac{|b|}{\kappa_1 m} \sqrt{1 + \frac{\kappa_1^2 m^2}{b^2}}. \quad (17)$$

A geometrical reasoning due to van der Woude [25] showed that there exist two and only two solutions to Eq. (13) passing through two given points \mathbf{x}_A and \mathbf{x}_B and having their concavity oriented toward the origin, provided that \mathbf{x}_A and \mathbf{x}_B are not aligned with O . Consequently, there exist two and only two time transfer functions in the background defined by the optical metric (8). Of course, the time transfer functions coincide when \mathbf{x}_A and \mathbf{x}_B are aligned with O .

As long as points \mathbf{x}_A and \mathbf{x}_B are located at finite distances from O , the range of φ is limited by the inequalities

$$-\frac{1}{e} < \cos(\varphi - \varphi_P) \leq 1. \quad (18)$$

In order to determine the time transfer functions, we may assume that φ_A and φ_B are given in such a way that inequalities as follows

$$0 < \varphi_B - \varphi_A < \pi \quad (19)$$

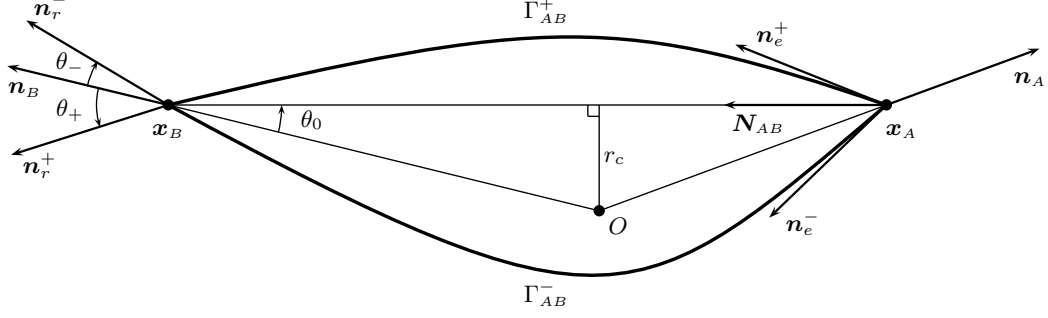


FIG. 1. The two possible light rays joining two given points within the linearized, weak-field approximation.

hold. We denote by Γ_{AB}^+ the trajectory of light joining \mathbf{x}_A and \mathbf{x}_B along which φ continuously increases from φ_A to φ_B (see Fig. 1). With our convention that $E > 0$, it follows from (11) and (12) that the parameter b has a value $b_+ > 0$.

There exists also a light ray along which φ continuously decreases from φ_A to $\bar{\varphi}_B$, with $\bar{\varphi}_B$ being defined by

$$\bar{\varphi}_B = \varphi_B - 2\pi. \quad (20)$$

The trajectory of this light ray will be denoted by Γ_{AB}^- . With our convention on the sign of E , we have $b_- < 0$.

The time transfer functions corresponding to Γ_{AB}^+ and to Γ_{AB}^- will be denoted by \mathcal{T}_+ and \mathcal{T}_- , respectively. As we shall see below, the explicit expression of these functions may be inferred from the expressions of b_+ and b_- as functions of r_A, r_B and $\varphi_B - \varphi_A$. The analytical determination of the quantities b_+ and b_- is the subject of the following section.

IV. DETERMINATION OF THE LIGHT RAYS PASSING BY TWO POINTS

It is straightforwardly inferred from (15) that e, p and φ_P must satisfy two relations as follows:

$$\frac{p}{er_A} - \frac{1}{e} = \cos(\varphi_A - \varphi_P), \quad (21a)$$

$$\frac{p}{er_B} - \frac{1}{e} = \cos(\varphi_B - \varphi_P). \quad (21b)$$

It is possible to eliminate the unknown quantity φ_P . Indeed, taking Eqs. (21) into account, we can form the system of equations

$$\begin{aligned} \left(\frac{p}{er_B} - \frac{1}{e}\right) \cos \varphi_A - \left(\frac{p}{er_A} - \frac{1}{e}\right) \cos \varphi_B \\ = \sin(\varphi_B - \varphi_A) \sin \varphi_P, \end{aligned} \quad (22a)$$

$$\begin{aligned} \left(\frac{p}{er_A} - \frac{1}{e}\right) \sin \varphi_B - \left(\frac{p}{er_B} - \frac{1}{e}\right) \sin \varphi_A \\ = \sin(\varphi_B - \varphi_A) \cos \varphi_P. \end{aligned} \quad (22b)$$

Squaring (22a) and (22b), and then adding the relations thus obtained, we get an equation independent of φ_P in which we can substitute for p and e from (16) and (17), respectively. Putting

$$\mu = \cos(\varphi_B - \varphi_A), \quad (23)$$

it is easily seen that this equation may be written in the form

$$\begin{aligned} b^4 - \frac{r_A^2 r_B^2}{R_{AB}^2} (1 - \mu) \left[1 + \mu + \frac{2\kappa_1 m(r_A + r_B)}{r_A r_B} \right] b^2 \\ + \kappa_1^2 m^2 \frac{r_A^2 r_B^2}{R_{AB}^2} (1 - \mu)^2 = 0, \end{aligned} \quad (24)$$

where R_{AB} is the usual Euclidean distance between \mathbf{x}_A and \mathbf{x}_B given by

$$R_{AB} = |\mathbf{x}_B - \mathbf{x}_A| = \sqrt{r_A^2 + r_B^2 - 2\mu r_A r_B}. \quad (25)$$

Considered as an equation for b^2 , Eq. (24) has two real roots b_+^2 and b_-^2 .

When $\mu = -1$, these roots coincide. Their shared value is then

$$b_+^2 = b_-^2 = 2\kappa_1 m \frac{r_A r_B}{r_A + r_B}. \quad (26)$$

When $-1 < \mu < 1$, the roots of Eq. (24) are given by

$$b_{\pm}^2 = \frac{r_A^2 r_B^2 (1 - \mu^2)}{2R_{AB}^2} \left[1 + \frac{2\kappa_1 m(r_A + r_B)}{r_A r_B (1 + \mu)} \right. \\ \left. \pm \sqrt{1 + \frac{4\kappa_1 m(r_A + r_B)}{r_A r_B (1 + \mu)} + \frac{8\kappa_1^2 m^2}{r_A r_B (1 + \mu)}} \right]. \quad (27)$$

Both b_+^2 and b_-^2 are positive. This property is a direct consequence of the expression of b_+^2 and of the following relation, inferred from (24):

$$b_+^2 b_-^2 = \kappa_1^2 m^2 \frac{r_A^2 r_B^2 (1 - \mu^2)}{R_{AB}^2}. \quad (28)$$

Of course, b_+ and b_- could be determined by taking the square root of the right-hand side of Eq. (27). How-

ever, the expression of these quantities can be notably simplified. Equation (24) yields

$$b_+^2 + b_-^2 = \frac{r_A^2 r_B^2 (1 - \mu^2)}{R_{AB}^2} \left[1 + \mu + \frac{2\kappa_1 m(r_A + r_B)}{r_A r_B} \right]. \quad (29)$$

Furthermore, Eq. (28) implies

$$b_+ b_- = -\kappa_1 m \frac{r_A r_B}{R_{AB}} (1 - \mu), \quad (30)$$

the minus sign in the right-hand side being due to the fact that b_+ and b_- have unlike signs. Then, calculating $(b_+ + b_-)^2$ and $(b_+ - b_-)^2$ from Eqs. (29) and (30) straightforwardly lead to expressions as follow for the constants of the motion b_+ and b_- :

$$b_{\pm} = \frac{1}{2} \frac{r_A r_B \sqrt{1 - \mu}}{R_{AB}} \left\{ \sqrt{1 + \mu + \frac{2\kappa_1 m(r_A + r_B - R_{AB})}{r_A r_B}} \pm \sqrt{1 + \mu + \frac{2\kappa_1 m(r_A + r_B + R_{AB})}{r_A r_B}} \right\}. \quad (31)$$

Knowing b_+ and b_- , the parameter p and the eccentricity e of the hyperbolas Γ_{AB}^+ and Γ_{AB}^- could be inferred from Eqs. (16) and (17), respectively. To complete the determination of the rays, we would still have to deduce the corresponding values of φ_P from Eqs. (21).

V. TIME TRANSFER FUNCTIONS

It follows from Eq. (43) given in [19] that the time transfer functions \mathcal{T}_+ and \mathcal{T}_- are linked to b_+ and b_- by the differential equations

$$b_{\pm} = -c\sqrt{1 - \mu^2} \frac{\partial \mathcal{T}_{\pm}}{\partial \mu}, \quad (32)$$

where b_{\pm} and \mathcal{T}_{\pm} are regarded as functions of r_A, r_B and μ . Integrating Eq. (32) leads to

$$c\mathcal{T}_{\pm}(r_A, r_B, \mu) = - \int_{\pm 1}^{\mu} \frac{b_{\pm}(r_A, r_B, \xi)}{\sqrt{1 - \xi^2}} d\xi \\ + c\mathcal{T}_{\pm}(r_A, r_B, \pm 1). \quad (33)$$

It is clear that $\mathcal{T}_+(r_A, r_B, 1) = \mathcal{T}_{rad}(r_A, r_B)$, where \mathcal{T}_{rad} is the radial time transfer function given by (9). So we

have

$$c\mathcal{T}_+(r_A, r_B, \mu) = \int_{\mu}^1 \frac{b_+(r_A, r_B, \xi)}{\sqrt{1 - \xi^2}} d\xi + c\mathcal{T}_{rad}(r_A, r_B). \quad (34)$$

The spherical symmetry of spacetime implies

$$c\mathcal{T}_-(r_A, r_B, -1) = c\mathcal{T}_+(r_A, r_B, -1).$$

Therefore, Eq. (33) for \mathcal{T}_- may be rewritten as

$$c\mathcal{T}_-(r_A, r_B, \mu) = - \int_{-1}^1 \frac{b_-(r_A, r_B, \xi)}{\sqrt{1 - \xi^2}} d\xi \\ + \int_{\mu}^1 \frac{b_-(r_A, r_B, \xi)}{\sqrt{1 - \xi^2}} d\xi + c\mathcal{T}_+(r_A, r_B, -1). \quad (35)$$

Inserting Eq. (31) into Eqs. (34) and (35) gives

$$c\mathcal{T}_+(r_A, r_B, \mu) = I_+(\mu) + I_-(\mu) \\ + c\mathcal{T}_{rad}(r_A, r_B), \quad (36a)$$

$$c\mathcal{T}_-(r_A, r_B, \mu) = I_+(-1) - I_-(-1) + I_-(\mu) \\ - I_+(\mu) + c\mathcal{T}_+(r_A, r_B, -1), \quad (36b)$$

where I_+ and I_- are defined as

$$I_{\pm}(\mu) = \frac{r_A r_B}{2} \int_{\mu}^1 \frac{1}{R(\xi)} \sqrt{1 + \frac{2\kappa_1 m[r_A + r_B \pm R(\xi)]}{r_A r_B (1 + \xi)}} d\xi, \quad (37)$$

with $R(\xi) = \sqrt{r_A^2 + r_B^2 - 2r_A r_B \xi}$. Since $R(\xi)$ is a strictly monotonic function of ξ , we can choose it as a new variable

for integrating Eqs. (37). We get thus

$$I_{\pm}(\mu) = \frac{1}{2} \int_{|r_B - r_A|}^{R_{AB}} \sqrt{\frac{r_A + r_B \mp R + 4\kappa_1 m}{r_A + r_B \mp R}} dR. \quad (38)$$

Performing the integration of Eqs. (38), and then taking into account Eq. (9) leads to expressions as follows for the time transfer functions:

$$c\mathcal{T}_{\pm}(\mathbf{x}_A, \mathbf{x}_B) = \frac{1}{2} \left(\sqrt{r_A + r_B + R_{AB}} \sqrt{r_A + r_B + R_{AB} + 4\kappa_1 m} \mp \sqrt{r_A + r_B - R_{AB}} \sqrt{r_A + r_B - R_{AB} + 4\kappa_1 m} \right) + 2\kappa_1 m \ln \left(\frac{\sqrt{r_A + r_B + R_{AB} + 4\kappa_1 m} + \sqrt{r_A + r_B + R_{AB}}}{\sqrt{r_A + r_B - R_{AB} + 4\kappa_1 m} \pm \sqrt{r_A + r_B - R_{AB}}} \right). \quad (39)$$

We note that Eqs. (39) yield expressions of the time transfer functions which are symmetric in \mathbf{x}_A and \mathbf{x}_B , as it could be expected.

When \mathbf{x}_A and \mathbf{x}_B are located in diametrically opposite directions, i.e. when $\mu = -1$, the expressions of $c\mathcal{T}_+$ and $c\mathcal{T}_-$ coincide and take the simple form

$$c\mathcal{T}_+ = c\mathcal{T}_- = \sqrt{r_A + r_B} \sqrt{r_A + r_B + 2\kappa_1 m} + 2\kappa_1 m \ln \left(\frac{\sqrt{r_A + r_B} + \sqrt{r_A + r_B + 2\kappa_1 m}}{\sqrt{2\kappa_1 m}} \right). \quad (40)$$

This formula gives the common value of the two time transfer functions for an Einstein ring.

VI. TWO STANDARD REGIMES RECOVERED

The radial variable takes its minimal value r_P when the derivative $dr/d\varphi = 0$. It is straightforwardly deduced from Eq. (13) that r_P^{\pm} satisfies the equation

$$(r_P^{\pm})^2 + 2\kappa_1 m r_P^{\pm} - b_{\pm}^2 = 0. \quad (41)$$

So, the values of r_P^+ and r_P^- are given by

$$r_P^{\pm} = \sqrt{b_{\pm}^2 + \kappa_1^2 m^2} - \kappa_1 m. \quad (42)$$

The calculations performed throughout this work are relevant only if $r \gg m$ at each point of the trajectory of the light ray under examination. Such a condition is obviously met for the Sun or the planets as long as the light ray is not bypassing the central mass. However, in almost all tests of gravitational theories and in gravitational lensing configurations, the light rays are skirting

around the central body, and then the condition

$$r_P^{\pm} \gg m \quad (43)$$

must hold. Taking Eq. (42) into account, this means that we must have

$$|b_{\pm}| \gg m. \quad (44)$$

So, we have to study the behavior of b_{\pm} and r_P^{\pm} when μ is close to -1 .

It is easily seen that if $\mu \rightarrow -1$, the derivative of b_{\pm}^2 with respect to μ has an asymptotic behavior given by

$$\frac{\partial b_{\pm}^2}{\partial \mu} \sim \pm \sqrt{\frac{\kappa_1 m}{1 + \mu} \left(\frac{r_A r_B}{r_A + r_B} \right)^3 \left(1 + \frac{2\kappa_1 m}{r_A + r_B} \right)}. \quad (45)$$

It follows from Eq. (45) that b_{\pm}^2 is an increasing function of μ when μ is sufficiently close to -1 . Therefore, b_{\pm}^2 reaches a local lower bound when $\mu = -1$, the value of which is given by Eq. (26). Taking into account Eq. (42), it may be inferred that r_P^+ reaches a minimum when $\mu = -1$, namely

$$(r_P^+)_{min} = \sqrt{\frac{2\kappa_1 m r_A r_B}{r_A + r_B} + \kappa_1^2 m^2} - \kappa_1 m. \quad (46)$$

Since it is assumed that $r_A \gg m$ and $r_B \gg m$, it follows from Eq. (46) that $(r_P^+)_{min} \gg m$. This means that the ray Γ_{AB}^+ is correctly described by the present formalism in the configurations of conjunction or quasiconjunction, i.e. when μ is in the vicinity of -1 . This conclusion can be extended to the other values of μ , in spite of the fact that b_{\pm} is not a monotonic function of μ .

Equation (45) shows that $|b_{-}|$ is a decreasing function when $\mu \rightarrow -1$. In fact, it may be seen that this property is valid on the range $-1 \leq \mu < 1$ since Eq. (31) for b_{-} may be rewritten in the form

$$b_{-} = - \frac{2\kappa_1 m \sqrt{1 - \mu}}{\sqrt{1 + \mu + \frac{2\kappa_1 m(r_A + r_B + R_{AB})}{r_A r_B}}} + \sqrt{1 + \mu + \frac{2\kappa_1 m(r_A + r_B - R_{AB})}{r_A r_B}}. \quad (47)$$

Indeed, the numerator $\sqrt{1 - \mu}$ in the right-hand side of

Eq. (47) is a decreasing function and the denominator

an increasing function. So, r_P^- reaches its upper bound $(r_P^-)_{max}$ when $\mu = -1$. This upper bound is equal to $(r_P^+)_{min}$. Hence

$$(r_P^-)_{max} = \sqrt{\frac{2\kappa_1 m r_A r_B}{r_A + r_B} + \kappa_1^2 m^2} - \kappa_1 m. \quad (48)$$

We are led to distinguish two regimes for the rays Γ_{AB}^- .

A. First regime

The value $\mu = -1$ corresponds to an Einstein ring. We infer from Eq. (26) that $|b_-| \gg m$, and consequently $(r_P^-)_{max} \gg m$ hold in this configuration. When μ is very close to -1 or varies so that to have

$$1 + \mu \propto 1 + \mu_E, \quad (49)$$

where μ_E is a typical value defined by

$$1 + \mu_E = \frac{\kappa_1 m (r_A + r_B)}{r_A r_B}, \quad (50)$$

the order of magnitude of $|b_-|$ is given by

$$|b_-| \propto \sqrt{\frac{\kappa_1 m r_A r_B}{r_A + r_B}}. \quad (51)$$

Consequently, the condition $r_P^- \gg m$ is met, which implies that the light ray Γ_{AB}^- is entirely located in the region where the weak field approximation is valid, as it is also the case for the ray Γ_{AB}^+ (see above). We can conclude that this regime corresponds to a configuration of gravitational lensing in a weak field, with two images of the same point-mass object. We will show in Sec. VIII that the well-known formulas of the weak gravitational lensing may be recovered from Eqs. (39).

B. Second regime

For the values of μ such that

$$2 > 1 + \mu \gg 1 + \mu_E, \quad (52)$$

Eq. (47) implies

$$|b_-| \approx \kappa_1 m \sqrt{\frac{1 - \mu}{1 + \mu}}. \quad (53)$$

The magnitude of $|b_-|$ is then of the order of $\kappa_1 m$. We have therefore $r_P^- \sim \kappa_1 m$, a value which is not compatible with the weak-field approximation. So, only the ray Γ_{AB}^+ can be treated with our method when the condition (52) is met. Configurations of this kind occur in the experiments currently performed in the solar system. We are then facing cases where only the time transfer function \mathcal{T}_+ has to be considered. We will see in Sec. IX how the Shapiro time delay formula supplemented by enhanced terms of any order is recovered from the expression of $c\mathcal{T}_+$ given by Eq. (39).

VII. DIRECTION OF LIGHT PROPAGATION

Let $x^\alpha = x_\Gamma^\alpha(\zeta)$ be a system of parametric equations describing a null geodesic Γ joining \mathbf{x}_A and \mathbf{x}_B , ζ being an arbitrarily chosen parameter. The direction of propagation at any point x of Γ is fully determined by the *direction triple* defined as [27]

$$\hat{\mathbf{k}} = (k_1/k_0, k_2/k_0, k_3/k_0), \quad (54)$$

where k_0 and k_i are the covariant components of the vector tangent to the ray at x , i.e. the quantities $k_\alpha = g_{\alpha\beta}(x_\Gamma^\beta(\zeta))dx_\Gamma^\beta(\zeta)/d\zeta$. This triple does not depend on the choice of the parameter along Γ .

In the optical metric (8), the vector $\hat{\mathbf{k}}$ is collinear to the 3-vector tangent to the photon trajectory since $\hat{\mathbf{k}} = -(1 + 2\kappa_1 m/r)d\mathbf{x}_\Gamma/dx^0$. It follows from this equation and from the property of k_α to be a null vector that the Euclidean norm of $\hat{\mathbf{k}}$ at any point x of Γ is given by

$$|\hat{\mathbf{k}}| = \sqrt{1 + 2\kappa_1 \frac{m}{r}}. \quad (55)$$

The propagation direction of light as seen by a static observer staying at x is therefore characterized by the unit *direction vector* \mathbf{n} defined as

$$\mathbf{n} = -\frac{\hat{\mathbf{k}}}{\sqrt{1 + 2\kappa_1 \frac{m}{r}}}. \quad (56)$$

The expressions of the direction vector at points \mathbf{x}_A and \mathbf{x}_B will be denoted by $\mathbf{n}_e(\mathbf{x}_A, \mathbf{x}_B)$ and $\mathbf{n}_r(\mathbf{x}_A, \mathbf{x}_B)$, respectively. These vectors can be derived from the relations

$$\hat{\mathbf{k}}_e(\mathbf{x}_A, \mathbf{x}_B) = c\nabla_{\mathbf{x}_A} \mathcal{T}_\Gamma(\mathbf{x}_A, \mathbf{x}_B), \quad (57a)$$

$$\hat{\mathbf{k}}_r(\mathbf{x}_A, \mathbf{x}_B) = -c\nabla_{\mathbf{x}_B} \mathcal{T}_\Gamma(\mathbf{x}_A, \mathbf{x}_B), \quad (57b)$$

where \mathcal{T}_Γ is the time transfer function associated to Γ and $\nabla_{\mathbf{x}}$ denotes the gradient with respect to \mathbf{x} (see [1] and Refs. therein).

We denote by \mathbf{n}^+ and \mathbf{n}^- the direction vectors relative to Γ_{AB}^+ and Γ_{AB}^- , respectively. Substituting for $c\mathcal{T}_\pm$ from Eq. (39) into Eqs. (57) yield expressions as follow:

$$\mathbf{n}_e^\pm(\mathbf{x}_A, \mathbf{x}_B) = \frac{1}{2} \frac{(X \pm Y)\mathbf{N}_{AB} - (X \mp Y)\mathbf{n}_A}{\sqrt{1 + 2\kappa_1 \epsilon_A}}, \quad (58a)$$

$$\mathbf{n}_r^\pm(\mathbf{x}_A, \mathbf{x}_B) = \frac{1}{2} \frac{(X \pm Y)\mathbf{N}_{AB} + (X \mp Y)\mathbf{n}_B}{\sqrt{1 + 2\kappa_1 \epsilon_B}}, \quad (58b)$$

where X and Y are defined by

$$X = \sqrt{1 + \frac{4\kappa_1 m}{r_A + r_B + R_{AB}}}, \quad (59)$$

$$Y = \sqrt{1 + \frac{4\kappa_1 m}{r_A + r_B - R_{AB}}}, \quad (60)$$

and

$$\mathbf{N}_{AB} = \frac{\mathbf{x}_B - \mathbf{x}_A}{|\mathbf{x}_B - \mathbf{x}_A|}, \quad \mathbf{n}_A = \frac{\mathbf{x}_A}{r_A}, \quad \mathbf{n}_B = \frac{\mathbf{x}_B}{r_B}, \quad (61)$$

$$\epsilon_A = \frac{m}{r_A}, \quad \epsilon_B = \frac{m}{r_B}. \quad (62)$$

The quantities X and Y may be transformed into

$$X = \sqrt{1 + \frac{2\kappa_1 m(r_A + r_B - R_{AB})}{r_A r_B (1 + \mu)}}, \quad (63)$$

$$Y = \sqrt{1 + \frac{2\kappa_1 m(r_A + r_B + R_{AB})}{r_A r_B (1 + \mu)}}. \quad (64)$$

Substituting for X and Y from Eqs. (63) and (64) into $X \pm Y$, and then comparing with Eq. (31) yield

$$X \pm Y = \frac{2b_{\pm}}{r_c}, \quad (65)$$

where r_c is the Euclidean distance of the straight line passing by \mathbf{x}_A and \mathbf{x}_B from the origin O , namely

$$r_c = \frac{r_A r_B \sqrt{1 - \mu^2}}{R_{AB}}. \quad (66)$$

Consequently, Eqs. (58) may be written in the form

$$\mathbf{n}_e^{\pm}(\mathbf{x}_A, \mathbf{x}_B) = \frac{b_{\pm} \mathbf{N}_{AB} - b_{\mp} \mathbf{n}_A}{r_c \sqrt{1 + 2\kappa_1 \epsilon_A}}, \quad (67a)$$

$$\mathbf{n}_r^{\pm}(\mathbf{x}_A, \mathbf{x}_B) = \frac{b_{\pm} \mathbf{N}_{AB} + b_{\mp} \mathbf{n}_B}{r_c \sqrt{1 + 2\kappa_1 \epsilon_B}}. \quad (67b)$$

Let us call θ_+ (resp. θ_-) the angle between the propagation direction of a radial light ray going to infinity through \mathbf{x}_B and the propagation direction of the light ray Γ_{AB}^+ (resp. Γ_{AB}^-) as measured by a static observer staying at point \mathbf{x}_B (see Fig. 1). Since we use an isotropic coordinate system adapted to the static character of the metric, it follows from a result shown in [9] that θ_+ (resp. θ_-) is equal to the Euclidean angle between \mathbf{n}_B and the direction vector \mathbf{n}_r^+ (resp. \mathbf{n}_r^-). The angles θ_+ and θ_- are regarded as algebraic quantities and their determination is chosen in such a way that $0 \leq |\theta_{\pm}| \leq \pi$, with the convention that θ_+ is a positive quantity. Comparing the exterior products $\mathbf{n}_B \times \mathbf{n}_r^+$ and $\mathbf{n}_B \times \mathbf{n}_r^-$ as derived from Eq. (67b), it appears that θ_+ and θ_- have unlike signs since the signs of b_+ and b_- are unlike. We have therefore an algebraic relation as follows:

$$\sin \theta_{\pm} = \frac{b_{\pm}}{r_B \sqrt{1 + 2\kappa_1 \epsilon_B}}. \quad (68)$$

To finish determining the values of θ_+ and θ_- , it is necessary to know the sign of $\cos \theta_{\pm}$. It follows from Eq. (67b) that

$$\cos \theta_{\pm} = \frac{b_{\pm} \cos \theta_0 + b_{\mp}}{r_c \sqrt{1 + 2\kappa_1 \epsilon_B}}, \quad (69)$$

where θ_0 is the angle between \mathbf{n}_B and \mathbf{N}_{AB} specified by the relations

$$\cos \theta_0 = \mathbf{n}_B \cdot \mathbf{N}_{AB} = \frac{r_B - r_A \mu}{R_{AB}}, \quad 0 \leq \theta_0 < \pi. \quad (70)$$

Equations (68)-(70) enable one to express the angles θ_+ and θ_- as functions of r_A , r_B and μ .

Taking into account Eqs. (29)-(30), and then noting that

$$\sin \theta_0 = |\mathbf{n}_B \times \mathbf{N}_{AB}| = \frac{r_A \sqrt{1 - \mu^2}}{R_{AB}}, \quad (71)$$

Eqs. (68)-(71) imply

$$\begin{aligned} \sin(\theta_+ + \theta_-) &= \sin \theta_0, \\ \cos(\theta_+ + \theta_-) &= \cos \theta_0. \end{aligned}$$

Hence the remarkable relation between θ_+ , θ_- and θ_0

$$\theta_+ + \theta_- = \theta_0. \quad (72)$$

This relation will be very useful for analyzing the lensing configurations, as it will be seen in the next section. It must be emphasized that Eq. (72) holds even if the angles θ_+ and θ_- are not small.

VIII. APPLICATION TO THE WEAK-FIELD GRAVITATIONAL LENSING

The general formulas of the previous section were established without being worried about the physical meaning of the rays joining \mathbf{x}_A and \mathbf{x}_B . So they may be applied to any configuration of gravitational lensing.

A. Angular separation between two images

The angular separation between the two images of the emitter as measured by a static observer staying at \mathbf{x}_B is the difference $\theta_+ - \theta_-$. The sine of $\theta_+ - \theta_-$ is easily inferred from the expression of the exterior product $\mathbf{n}_r^+ \times \mathbf{n}_r^-$. Using Eqs. (67b), and then taking into account Eqs. (27), we get

$$\begin{aligned} \sin(\theta_+ - \theta_-) &= \frac{r_A \sqrt{1 - \mu}}{R_{AB}(1 + 2\kappa_1 \epsilon_B)} \\ &\times \sqrt{1 + \mu + \frac{4\kappa_1 m(r_A + r_B)}{r_A r_B} + \frac{8\kappa_1^2 m^2}{r_A r_B}}. \end{aligned} \quad (73)$$

Let us apply the previous results to the case of an Einstein ring, which is characterized by $\theta_0 = 0$, i.e. $\mu = -1$. Denote by $2\theta_E$ the corresponding angular separation between the two images. We get from Eq. (73)

$$\sin 2\theta_E = \frac{2}{1 + 2\kappa_1 \epsilon_B} \sqrt{\frac{2\kappa_1 \epsilon_B r_A}{r_A + r_B} \left(1 + \frac{2\kappa_1 m}{r_A + r_B}\right)}. \quad (74)$$

Equation (74) is equivalent to a relation as follows

$$\sin \theta_E = \frac{1}{\sqrt{1 + 2\kappa_1 \epsilon_B}} \sqrt{\frac{2\kappa_1 \epsilon_B r_A}{r_A + r_B}}, \quad (75)$$

$$\sin(\theta_+ - \theta_-) = \sqrt{\frac{\sin^2 \theta_0}{(1 + 2\kappa_1 \epsilon_B)^2} + \frac{1 - \mu}{2} \left[1 + \frac{2r_A r_B (1 + \mu)}{R_{AB}^2} \right] \sin^2 2\theta_E}. \quad (76)$$

In the realistic lensing configurations, $1 + \mu$ is very close to 0 or varies in such way that $1 + \mu \propto 1 + \mu_E$, with μ_E defined by Eq. (50). Then θ_0 is small and $2r_A r_B (1 + \mu)/R_{AB}^2 \ll 1$. So θ_+ and θ_- are small and Eqs. (72) and (76) enable one to recover the well-known expressions of θ_+ and θ_- in lensing configurations, namely

$$\theta_{\pm} \approx \frac{\theta_0}{2} \pm \sqrt{\frac{\theta_0^2}{4} + \theta_E^2}. \quad (77)$$

Accordingly, Eq. (75) reduces to the usual expression of half the angular diameter of the Einstein ring, that is

$$\theta_E \approx \sqrt{\frac{2\kappa_1 m r_A}{(r_A + r_B) r_B}}, \quad (78)$$

and Eq. (71) shows that θ_0 is linked to $1 + \mu$ by a relation

which is straightforwardly inferred from Eqs. (26) and (68) when the equalities $\theta_+ = -\theta_- = \theta_E$ are used.

Taking into account Eq. (71), a short calculation shows that Eq. (73) may be rewritten in the form

as follows

$$\theta_0 \approx \frac{r_A}{r_A + r_B} \sqrt{2(1 + \mu)}. \quad (79)$$

We have therefore $\theta_0 \propto \theta_E$ when $1 + \mu \propto 1 + \mu_E$. We recover the crucial role played by the quantity μ_E defined by Eq. (50) in delimiting the lensing regime.

B. Difference in light travel times

The difference in light travel times $\mathcal{T}_- - \mathcal{T}_+$ is relevant since this quantity is observable. We obtain from Eqs. (39)

$$c(\mathcal{T}_- - \mathcal{T}_+) = \sqrt{r_A + r_B - R_{AB}} \sqrt{r_A + r_B - R_{AB} + 4\kappa_1 m} + 4\kappa_1 m \ln \left(\frac{\sqrt{r_A + r_B - R_{AB}} + \sqrt{r_A + r_B - R_{AB} + 4\kappa_1 m}}{2\sqrt{\kappa_1 m}} \right). \quad (80)$$

Since the argument of the logarithm in the right-hand side of Eq. (80) is always greater than or equal to 1, $\mathcal{T}_- - \mathcal{T}_+ \geq 0$ for any lensing configuration.

An expression of $\mathcal{T}_- - \mathcal{T}_+$ in terms of the observable angles θ_+ and θ_- is possible. Taking into account the fact that Eq. (60) implies

$$r_A + r_B - R_{AB} = \frac{4\kappa_1 m}{Y^2 - 1}, \quad (81)$$

it may be seen that Eq. (80) reads

$$c(\mathcal{T}_- - \mathcal{T}_+) = \frac{4\kappa_1 m Y}{Y^2 - 1} + 4\kappa_1 m \ln \sqrt{\frac{Y + 1}{Y - 1}}. \quad (82)$$

It follows from Eqs. (65), (66) and (68) that

$$Y = \sqrt{1 + 2\kappa_1 \epsilon_B} \frac{R_{AB}}{r_A \sqrt{1 - \mu^2}} (\sin \theta_+ - \sin \theta_-).$$

Taking into account Eqs. (71) and (72), this equation may be written as

$$Y = \sqrt{1 + 2\kappa_1 \epsilon_B} \frac{\sin \theta_+ - \sin \theta_-}{\sin(\theta_+ + \theta_-)}. \quad (83)$$

Substituting for Y from Eq. (83) into Eq. (82), and then noting that $Y > 1$ and that $\sin(\theta_+ + \theta_-) > 0$ follows from Eqs. (70) and (72), it is easily checked that the difference in light travel time due to the gravitational lensing may be written in the form

$$c(\mathcal{T}_- - \mathcal{T}_+) = 4\kappa_1 m \frac{\sqrt{1 + 2\kappa_1 \epsilon_B} (\sin \theta_+ - \sin \theta_-) \sin(\theta_+ + \theta_-)}{(1 + 2\kappa_1 \epsilon_B) (\sin \theta_+ - \sin \theta_-)^2 - \sin^2(\theta_+ + \theta_-)} + 2\kappa_1 m \ln \left[\frac{\sqrt{1 + 2\kappa_1 \epsilon_B} (\sin \theta_+ - \sin \theta_-) + \sin(\theta_+ + \theta_-)}{\sqrt{1 + 2\kappa_1 \epsilon_B} (\sin \theta_+ - \sin \theta_-) - \sin(\theta_+ + \theta_-)} \right]. \quad (84)$$

As it has been pointed out in the discussion of the apparent angles, the terms of order ε_B in Eq. (84) can be neglected and the angles θ_+ and θ_- are very small in astrophysical observations of double images. In practice, Eq. (84) reduces therefore to

$$c(\mathcal{T}_- - \mathcal{T}_+) \approx \kappa_1 m \frac{\theta_+^2 - \theta_-^2}{|\theta_+ \theta_-|} + 2\kappa_1 m \ln \left| \frac{\theta_+}{\theta_-} \right|. \quad (85)$$

In general relativity, Eq. (85) coincides with the formula (4.78a) in [26] giving the difference in light travel time due to the gravitational lensing when the cosmological redshift is neglected.

IX. SHAPIRO TIME DELAY AND ENHANCED TERMS

After some calculations, the function $c\mathcal{T}_+$ given by Eq. (39) may be rewritten in the form

$$\begin{aligned} c\mathcal{T}_+ = & R_{AB} + \kappa_1 m \ln \left(\frac{r_A + r_B + R_{AB}}{r_A + r_B - R_{AB}} \right) \\ & + \frac{8\kappa_1^2 m^2 R_{AB}}{r_A r_B (1 + \mu)} \frac{1}{(1 + X)(1 + Y)(X + Y)} \\ & + 2\kappa_1 m \ln \left(\frac{1 + X}{1 + Y} \right). \end{aligned} \quad (86)$$

The two first terms in the right-hand side of Eq. (86) are the Euclidean distance between \mathbf{x}_A and \mathbf{x}_B supplemented by the Shapiro time delay [16, 17]. Substituting Eqs. (63) and (64) into Eq. (86), and then performing a Taylor expansion about $m = 0$ yields

$$c\mathcal{T}_+ = R_{AB} + \kappa_1 m \ln \left(\frac{r_A + r_B + R_{AB}}{r_A + r_B - R_{AB}} \right) + c \sum_{n=2}^{\infty} \mathcal{T}_+^{(n)}, \quad (87)$$

where each function $\mathcal{T}_+^{(n)}$ may be written in a form as $\kappa_1^n m^n \Theta_+^{(n)}(r_A, r_B, \mu)/(1 + \mu)^{n-1}$, $\Theta_+^{(n)}$ being a regular function when $\mu \rightarrow -1$. Consequently, in this regime, Γ_{AB}^+ is a quasi-Minkowskian light ray of the metric (8).

To determine the values of μ for which this expansion is convergent, it is convenient to return to Eq. (36). It follows from (9) and (38) that \mathcal{T}_{rad} and $I_-(\mu)$ can be developed in power series of m whatever μ . In contrast, it is easily seen that the integrand in Eq. (38) written for $I_+(\mu)$ can be expanded in a power series in m for any R in the range $[|r_B - r_A|, R_{AB}]$ if and only if $R_{AB} \leq r_A + r_B - 4\kappa_1 m$, a condition which is equivalent to

$$1 + \mu \geq \frac{4\kappa_1 m(r_A + r_B - 2\kappa_1 m)}{r_A r_B}. \quad (88)$$

Consequently, $I_+(\mu)$ can be expanded as a power series if the condition (88) is met. This feature and the fact that the quantity Y involved in Eq. (86) cannot be expanded

in a power series in m if $R_{AB} > r_A + r_B - 4\kappa_1 m$ enable one to conclude that the expansion in Eq. (87) is convergent if and only if the inequality (88) is satisfied.

In practice, it can be said that the Taylor series (87) is convergent if $1 + \mu \geq 4(1 + \mu_E)$, *a fortiori* if the inequalities (52) are satisfied. For $n = 2, \dots, 5$, the functions $\mathcal{T}_+^{(n)}$ are given by

$$c\mathcal{T}_+^{(2)}(\mathbf{x}_A, \mathbf{x}_B) = -\kappa_1^2 m^2 \frac{R_{AB}}{r_A r_B (1 + \mu)}, \quad (89)$$

$$c\mathcal{T}_+^{(3)}(\mathbf{x}_A, \mathbf{x}_B) = \kappa_1^3 m^3 \frac{R_{AB}(r_A + r_B)}{r_A^2 r_B^2 (1 + \mu)^2}, \quad (90)$$

$$c\mathcal{T}_+^{(4)}(\mathbf{x}_A, \mathbf{x}_B) = -\frac{5}{12} \kappa_1^4 m^4 \frac{R_{AB}[3(r_A + r_B)^2 + R_{AB}^2]}{r_A^3 r_B^3 (1 + \mu)^3}, \quad (91)$$

$$c\mathcal{T}_+^{(5)}(\mathbf{x}_A, \mathbf{x}_B) = \frac{7}{4} \kappa_1^5 m^5 \frac{R_{AB}(r_A + r_B)[(r_A + r_B)^2 + R_{AB}^2]}{r_A^4 r_B^4 (1 + \mu)^4}. \quad (92)$$

Let us compare these results with the known terms in the right-hand side of Eq. (3) which correspond to the metric defined by Eqs. (5) and (6). We have for $\mathcal{T}^{(2)}$ (see [1, 5, 18]) and $\mathcal{T}^{(3)}$ (see [19, 20])

$$c\mathcal{T}^{(2)}(\mathbf{x}_A, \mathbf{x}_B) = -m^2 \frac{R_{AB}}{r_A r_B} \left(\frac{\kappa_1^2}{1 + \mu} - \kappa_2 \frac{\arccos \mu}{\sqrt{1 - \mu^2}} \right), \quad (93)$$

$$\begin{aligned} c\mathcal{T}^{(3)}(\mathbf{x}_A, \mathbf{x}_B) = & m^3 \frac{R_{AB}(r_A + r_B)}{r_A^2 r_B^2 (1 + \mu)} \\ & \times \left(\frac{\kappa_1^3}{1 + \mu} - \kappa_1 \kappa_2 \frac{\arccos \mu}{\sqrt{1 - \mu^2}} + \kappa_3 \right), \end{aligned} \quad (94)$$

where κ_2 and κ_3 are coefficients appearing in Eq. (6).

The function $\mathcal{T}_+^{(2)}$ yielded by Eq. (89) corresponds to the so-called “enhanced term” which appears in the right-hand side of Eq. (93) (see [2, 9, 18]). This term is usually regarded as worrying on the grounds of its unboundedness when $1 + \mu \rightarrow 0$. It must be emphasized, however, that $\mathcal{T}_+^{(2)}$ is relevant only when the condition in (88) is satisfied. As a consequence, the divergence of $\mathcal{T}^{(2)}$ when $1 + \mu \rightarrow 0$ has no physical meaning, a feature which is confirmed by the property of the function \mathcal{T}_+ to be regular on the range $0 \leq 1 + \mu \leq 2$. The same analysis holds for $\mathcal{T}_+^{(3)}$, which corresponds to the enhanced term proportional to $\kappa_1^3 m^3/(1 + \mu)^2$ occurring in the term $\mathcal{T}^{(3)}$ given by Eq. (94). So, it is natural to conjecture that the dominant enhanced term appearing in the perturbation function $\mathcal{T}^{(n)}$ corresponds to the term $\mathcal{T}_+^{(n)}$ for any n .

To finish, let us note that the expansion

$$\kappa_1 m \ln \left(\frac{r_A + r_B + R_{AB} + \kappa_1 m}{r_A + r_B - R_{AB} + \kappa_1 m} \right) = \kappa_1 m \ln \left(\frac{r_A + r_B + R_{AB}}{r_A + r_B - R_{AB}} \right) + c \left(\mathcal{T}_+^{(2)} + \frac{1}{2} \mathcal{T}_+^{(3)} + \frac{1}{5} \mathcal{T}_+^{(4)} + \frac{1}{14} \mathcal{T}_+^{(5)} + \dots \right) \quad (95)$$

is valid when inequality (88) holds. As a consequence, Eq. (87) may be rewritten as an expansion from which the term $\mathcal{T}_+^{(2)}$ is removed:

$$c\mathcal{T}_+ = R_{AB} + \kappa_1 m \ln \left(\frac{r_A + r_B + R_{AB} + \kappa_1 m}{r_A + r_B - R_{AB} + \kappa_1 m} \right) + c \left(\frac{1}{2} \mathcal{T}_+^{(3)} + \frac{4}{5} \mathcal{T}_+^{(4)} + \frac{13}{14} \mathcal{T}_+^{(5)} + \dots \right). \quad (96)$$

If one neglects the terms of order higher than two, Eq. (96) reduces to the expression of the time transfer function introduced by Moyer on the basis of a rather difficult reasoning [21]. So Moyer's formula incorporates the dominant term in Eq. (93). This is the reason for using Moyer's expression instead of Shapiro's formula in the discussions of optical effects in the solar system as long as the third-order term $\mathcal{T}_+^{(3)}$ may be neglected (see, e.g., Refs. [2–4, 6]).

X. CONCLUSION

In this paper we have developed a complete description of light rays within the linearized, weak-field approximation of the Schwarzschild-like metrics. Our main result is the explicit computation of the two possible time transfer functions \mathcal{T}_+ and \mathcal{T}_- relative to the optical metric (8). The expressions of these functions yielded by Eq. (39) are valid for any points \mathbf{x}_A and \mathbf{x}_B , provided that

the corresponding light rays are confined in the region $r \gg m$.

Our analysis leads to distinguish two standard regimes. The first regime corresponds to gravitational lensing configurations in which two images of a given source can be observed. The second regime corresponds to the cases where the light ray associated with \mathcal{T}_- has an impact parameter of the order of m , which implies that only the ray corresponding to \mathcal{T}_+ has to be kept in our scheme.

We carry out the calculation of the propagation direction of each possible ray. Then, for gravitational lensing configurations, we find the general expressions yielding the angular separation and the difference in light travel time between the two images of a source. When the angles are very small, our formulas reduce to the usual relations for the static, spherically symmetric lensing.

In the second regime, we expand the function \mathcal{T}_+ as a Taylor series in powers of m . It appears that this series is convergent if and only if the condition in Eq. (88) is met. Explicit calculations of the terms involved in the expansion are carried out up to the fifth order. The first-order term is the Shapiro time delay. A comparison of the second- and third-order terms with Eqs. (93) and (94) strongly suggests that $\mathcal{T}_+^{(n)}$ corresponds to the dominant enhanced term in $\mathcal{T}^{(n)}$ for any n . As a final remark, we point out that the second-order term we find in the expansion of \mathcal{T}_+ can be absorbed in the logarithmic function proposed by Moyer, as it is shown by Eq. (96).

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